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USING SVD TO DETECT DAMAGE IN STRUCTURES WITH DIFFERENT OPERATIONAL CONDITIONS

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The aim of this article is to present a method, based on the use of singular value decomposition (SVD), for detecting damage in structures at an early stage. The main advantage of the proposed technique is firstly the capability to deal with structures with different operating conditions, i.e. with dynamic characteristics which may change during the lifetime. Secondly, no mathematical model of the analyzed structure is required *a priori*, such that it can be applied to systems of arbitrary complexity. Application of this method to simulated data for a truss-type structure is illustrated, with results from an experimental investigation on a cantilever beam.

1. INTRODUCTION

The ability to detect damage in mechanical, aerospace and civil structures is becoming increasingly important and, as a consequence, wide variety of non-destructive evaluation methods are being developed. Most of the techniques currently available aim to detect small defects close to the surface of the structure and near the sensor position. Vibration-based inspection is currently receiving increased attention, mainly due to the possibility of detecting faults at unmeasured locations by monitoring, during the lifetime, the dynamic characteristics of the structure under test. The concept behind this type of technique is based on the analysis of the dynamic response and variations in the spectra [1–3].

Even though the problem of locating and assessing the severity of damage is important in terms of estimating the residual life of the structure, reliable detection of the occurrence of damage is arguably the most important aspect. Indeed, if damage could be detected, it may be feasible to take the structure off-line and perform an extensive investigation in order to determine the location and extent of

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the fault. As a consequence the aim of the research described in this article has been to develop a fundamental level method (level 1 according to Rytter's classification of damage detection techniques [4]), to permit diagnosis of damage in structures of arbitrary complexity by comparing the current dynamic response characteristics with measurements made previously at the same locations.

A number of studies [5–8] deal with the detection of damage by using shifts in measured natural frequency caused by a fault. However, it should be noted that frequency shift-based techniques may have limited application for the detection of damage in real structures. Indeed, in several situations the structure under test may alter during normal operational conditions, e.g., off-shore platforms subject to tidal variations in sea level or changing mass due to storage of oil. With concrete bridges, for example, changes in natural frequencies have been correlated to temperature variations of the bridge deck during the hours of the day [9–12]. In these cases, a fundamental level technique should be able to detect the onset of damage and distinguish it from a change in the operational or environmental conditions. This problem has been addressed by Worden *et al.* [13–15], who demonstrated the possibility of performing this task by taking advantage of the classification properties of neural networks.

The aim of this paper is to summarize and extend previous works by Ruotolo and Surace [16, 17], in which a new method, based on the use of singular value decomposition, was proposed in order to detect structural damage. In the past this numerical method has been used for several applications in structural dynamics, being particularly useful for matrix inversion, required in several identification procedures, as outlined in references [18–21].

The damage detection method proposed in this paper as well as novelty detection presented by Worden [13] is capable of diagnosing damage even if the structure under test can modify its dynamic behaviour due to change in operational condition. To demonstrate that the method can detect damage at an early stage data obtained by using a finite element model of a truss structure and also by performing experimental vibration tests on a notched cantilever beam with concentrated masses of different weight attached close to the free end have been analyzed.

2. USE OF SINGULAR VALUE DECOMPOSITION (SVD) TO DETECT DAMAGE

2.1. DESCRIPTION OF THE METHOD

The damage detection method proposed in this article is based on the determination of the rank of a matrix. The specific objective is to detect structural damage even if the structure monitored operates normally in a range of n different conditions, such that the dynamic response of the structure will change while remaining undamaged.

For each operating condition both in undamaged and damaged states, it is possible to introduce a characteristic vector $\mathbf{w}$ in which a particular property of the structure is represented, e.g., individual natural frequencies, mode shapes, frequency response functions, transmissibilities, etc. Furthermore, $\mathbf{w}_i$ is related to the normal operational condition i, while $\mathbf{w}_c$ is related to the current operational condition, in which the structure may be damaged.

Due to the effect of damage on the dynamic behaviour of the inspected structure, it is assumed that a fault will alter the vibrational response and, as a consequence, also the characteristic vectors. A straightforward way of solving the problem of damage detection is to compare characteristic vectors $\mathbf{w}_i$ and $\mathbf{w}_c$, in order to discover any significant deviation which cannot be attributed to noise, for each of the different operational conditions, i.e., verify that

$$\mathbf{w}_c \neq \mathbf{w}_i \quad \forall i \in [1, n]. \tag{1}$$

The characteristic vectors $\mathbf{w}$ can be arranged in a matrix $\mathbf{M}$ in the following way,

$$\mathbf{M} = [\mathbf{w}_1 \;\; \mathbf{w}_2 \cdots \mathbf{w}_n \;\; \mathbf{w}_c],$$

such that condition (1) can be checked by estimating the rank of the matrix $\mathbf{M}$ in order to determine whether or not a structure is damaged. Indeed, if the structure is undamaged, the characteristic vector $\mathbf{w}_c$ will be equal to one of the characteristic vectors $\mathbf{w}_1 \ldots \mathbf{w}_n$ and the rank of the matrix $\mathbf{M}$ will be equal to n. On the other hand, if the structure under test is damaged, the rank of the matrix $\mathbf{M}$ will be equal to $n + 1$.

Since it is usual to use different types of sensors during monitoring of real structures, e.g., accelerometers and displacement transducers [11], it may be advantageous to employ a technique capable of incorporating and comparing different kinds of data in order to detect the onset of damage. The method presented in this article can be extended simply to deal with this situation, i.e., to cases in which p characteristic vectors (of the same dimension) can be associated to each operational condition. As a consequence, a characteristic matrix $\mathbf{W}_i$ with p columns can be introduced, where the jth column is a previously defined characteristic vector $\mathbf{w}_i^{(j)}$. By following the same procedure outlined previously, matrix $\mathbf{M}$ will be

$$\mathbf{M} = [\mathbf{W}_1 \;\; \mathbf{W}_2 \cdots \mathbf{W}_n \;\; \mathbf{W}_c],$$

which has $p \cdot (n + 1)$ columns. In this case, in order to determine the rank variation of matrix $\mathbf{M}$ due to the onset of damage, it is necessary to evaluate beforehand the rank of characteristic matrices $\mathbf{W}_i$. If different measurements are incorporated in $\mathbf{W}_i$, all columns could be linearly independent, such that the rank of each $\mathbf{W}_i$ will be equal to p; instead, if all columns of characteristic matrix $\mathbf{W}_i$ are linearly dependent, its rank will be equal to 1. With m used to denote the number of linearly independent columns in the characteristic matrix $\mathbf{W}_i$ ($1 \leqslant m \leqslant p$) the rank of the matrix $\mathbf{M}$ will be $m \cdot n$, when the structure is undamaged. If damaged the rank of $\mathbf{M}$ will increase.

Thus, to assess the integrity of a structure, it is necessary to determine the rank of matrix $\mathbf{M}$, which is constructed after acquiring and analyzing the dynamic response of the structure. Usually, this computation is performed via Gaussian elimination but, in practice, the rows of a matrix are neither exactly orthogonal nor parallel, so that it may be difficult to estimate the rank after performing this numerical procedure. Moreover, the numerical technique used to evaluate the rank of the matrix $\mathbf{M}$ should be able to detect near rank deficiency which occurs for small scale damage, as the vector $\mathbf{w}_c$ (or matrix $\mathbf{W}_c$) is "near" to the corresponding vector $\mathbf{w}_i$ (or matrix $\mathbf{W}_i$) of the undamaged structure in the same operational condition. In such situations, it is well-known that SVD represents a sensitive means of evaluating the rank of a matrix [23].

The aim of SVD is to decompose a given matrix $\mathbf{A}$ into the product

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}}, \tag{2}$$

where $\mathbf{U}$ and $\mathbf{V}$ are two orthonormal matrices, i.e., $\mathbf{U}^{\mathrm{H}}\mathbf{U} = \mathbf{I}$ and $\mathbf{V}^{\mathrm{H}}\mathbf{V} = \mathbf{I}$, and $\mathbf{\Sigma}$ contains along the diagonal the singular values σ of the matrix $\mathbf{A}$ sorted in a descending order. If the matrix $\mathbf{A}$ is rank deficient, i.e., some rows (or columns) can be generated by a linear superposition of the others, some of the singular values will be zero [22, 23]. Accordingly, SVD can be used to estimate the rank of a matrix by simply evaluating the number of singular values which are not zero and which cannot be related to the measurement noise.

2.2. INFLUENCE OF MEASUREMENT NOISE

When dealing with experimental data it is necessary to take into account the influence of measurement noise and its effect on the determination of the rank of matrix $\mathbf{M}$.

In order to separate the data related to the structure from the obscuring noise it is advisable to introduce into matrix $\mathbf{M}$ more characteristic matrices $\mathbf{W}_i$ for each operational condition; this procedure could be seen as an average of the various measurements performed on the structure under test at the same condition. By collecting l measurements for each one of the n conditions, matrix $\mathbf{M}$ will be

$$\mathbf{M} = [\mathbf{W}_1^{(1)}\mathbf{W}_1^{(2)}\cdots\mathbf{W}_1^{(l)}\mathbf{W}_2^{(1)}\mathbf{W}_1^{(2)}\cdots\mathbf{W}_2^{(l)}\cdots\mathbf{W}_n^{(1)}\mathbf{W}_n^{(2)}\cdots\mathbf{W}_n^{(l)}\mathbf{W}_c^{(1)}\mathbf{W}_c^{(2)}\cdots\mathbf{W}_c^{(l)}].$$

Indeed, if the difference between two characteristic matrices $\mathbf{W}_i^{(j)}$ and $\mathbf{W}_i^{(k)}$ depends only on measurement noise, the rank of the matrix $\mathbf{M}_n$ will be equal to the rank of $\mathbf{M}$, and this fact can be used for detecting a fault.

Often measurement noise on experimental data can be considered to be either additive of multiplicative, or alternatively a combination of the two. In both cases, all the singular values of the matrix will be greater than zero [24].

In the presence of additive or *white* noise, which is uniform with the frequency and not correlated with the data, in order to distinguish between singular values related to the noise and those related to the state of the structure, a threshold level for the singular values related to the noise is introduced, as shown in Figure 1, such that any singular value greater than this threshold can be related to the state of the structure.

When performing this task periodically, an increase of the number of singular values over a predetermined threshold level should give an indication that the structure under investigation has altered its dynamic behaviour due to damage. Since any change of operating conditions should not alter this number of singular values, provided characteristic vectors corresponding to each operational

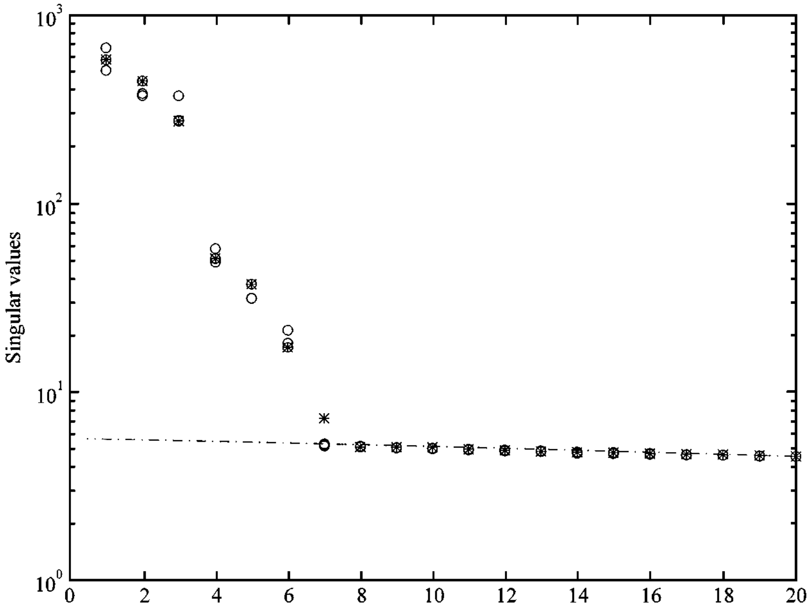

Figure 1. First 20 singular values in decreasing order for each undamaged operating condition and with damage level of 10% (additive noise), ○, undamaged; *, damaged.

condition have been included in the characteristic matrix, the method should be insensitive to a switching between different operating conditions.

In the case of multiplicative or *coloured* noise, for which peaks in the noise correspond to peaks in the data, it is impossible to introduce a simple threshold level which permits singular values related to the state of the structure to be distinguished from singular values related to the noise, due to the fact that the noise is correlated with the data (see Figure 2). Consequently, an improved damage index is needed to detect damage when applied to data corrupted by multiplicative noise.

2.3. FORMULATION OF A GENERAL DAMAGE INDEX

To address the difficulties associated with detecting damage on data influenced by noise which is not simply additive in nature, it is possible to take advantage of a property of singular values [25] of matrix **M** in order to formulate the damage index

$$D = \alpha \left(\prod_{k=1}^{r+1} \sigma_k(\mathbf{M}) \right), \tag{3}$$

where **M** is the matrix with multiple measurements for each operational condition, $\sigma_k(\mathbf{M})$ is the kth singular value of this matrix, r can be defined as the effective rank of the experimentally determined matrix **M** and α is a constant appropriately selected to normalize the index. The rank of **M** corresponds to the number of singular

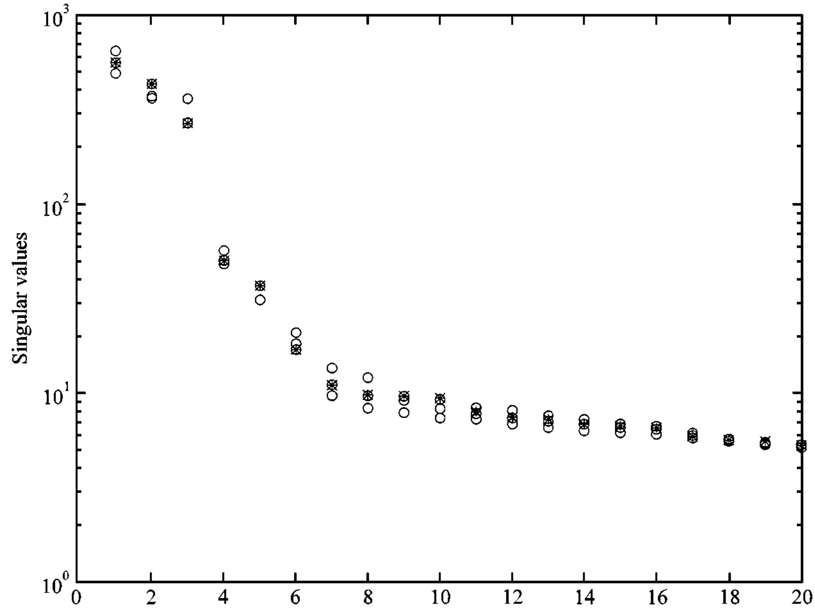

Figure 2. First 20 singular values in decreasing order for each undamaged operating condition and with damage level of 10% (multiplicative noise). ○, undamaged; *, damaged.

values which can be distinguished above the noise and therefore related to the undamaged structure.

As illustrated subsequently, the main characteristics of the damage index D are that, firstly, it does not vary by switching to a different normal operational condition [25] and, secondly, for a given condition it increases when the structure is damaged.

3. APPLICATION OF THE METHOD

3.1. ANALYSIS OF NUMERICAL DATA

The damage detection method described in the previous section has been applied to a truss structure, with dimensions as shown in Figure 3 and the following properties: Young's modulus $E = 7{\cdot}3 \times 10^{10}\,\mathrm{N/m^2}$, mass density $\rho = 2700\,\mathrm{kg/m^3}$, cross-section (for each rod) with width $b = 0{\cdot}02\,\mathrm{m}$ and height $h = 0{\cdot}02\,\mathrm{m}$. Moreover, the three following different mass conditions have been considered: (1) without any non-structural mass; (2) with a non-structural mass of 1 kg, concentrated at node 8; (3) with two non-structural masses of 1 kg each, concentrated at the nodes 8 and 11. The characteristic vectors have been built by using the amplitude of the frequency response functions (FRFs) $H^{(1)}(\omega)$ and $H^{(2)}(\omega)$ (shown in Figure 4, respectively, for normal conditions 1, 2 and 3) of the structure related, respectively, to the responses $y^{(1)}(t)$ and $y^{(2)}(\mathrm{t})$ and to an input force applied at node 7. As a consequence, a characteristic matrix with 2 columns

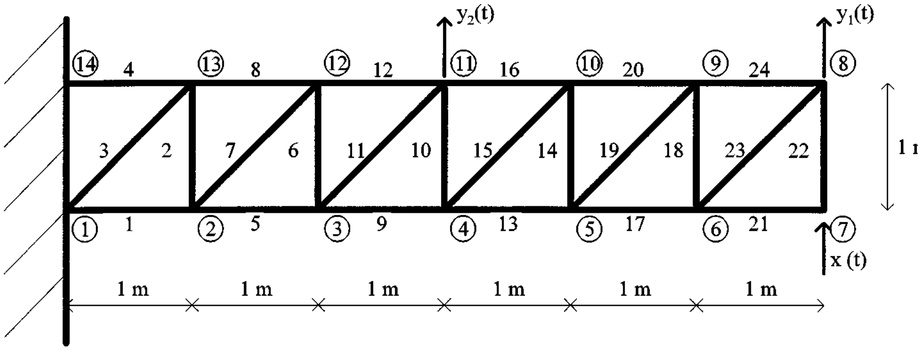

Figure 3. The examined truss structure.

$\mathbf{W}_i$ can be defined as

$$\mathbf{W}_i = [\mathbf{H}_i^{(1)}\mathbf{H}_i^{(2)}],$$

where each $\mathbf{H}_i^{(k)}$ is a vector obtained by evaluating the amplitude of $H_i^{(k)}(\omega)$ at 521 equallly spaced frequency points in the range from 85 to 215 Hz.

Figure 4 shows that in the frequency range under investigation there are two modes for each operational condition. As a result, each characteristic matrix $\mathbf{W}_i$ has two independent vectors, such that its rank is $m = 2$.

Matrix $\mathbf{M}$ for this structure and three operating conditions will be

$$\mathbf{M} = [\mathbf{H}_1^{(1)}\mathbf{H}_1^{(2)}\mathbf{H}_2^{(1)}\mathbf{H}_2^{(2)}\mathbf{H}_3^{(1)}\mathbf{H}_3^{(2)}\mathbf{H}_c^{(1)}\mathbf{H}_c^{(2)}],$$

and as a consequence matrix $\mathbf{M}$ has a dimension of 521×8. Moreover, since $m = 2$ and three operational conditions are analyzed, the rank r of matrix $\mathbf{M}$ will be equal to 6 if in the current condition the structure is undamaged.

Three levels of damage have been considered in which the stiffness of element 4 (indicated in Figure 3) has been reduced by 10, 30 and 50%. To investigate the capacity of this technique to detect damage, multiplicative noise (which is the more difficult type of noise to deal with) generated from a Gaussian distribution with standard deviation equal to 10% of the mean value of the matrix $\mathbf{M}$ was introduced onto the FRFs. However, in order to average the effect of noise on the results, it was decided to simulate 10 acquisitions of each FRF, such that matrix $\mathbf{M}$ has the form introduced in section 2.2 (in the following discussion the number of columns of $\mathbf{M}$ is multiplied by 10 as a consequence).

In order to demonstrate an application to damage detection, the case in which the undamaged truss structure goes through condition 1 to 2 and 3 is illustrated in Figure 5; in addition, the structure returns to condition 1 and then undergoes an increasing level of damage, from 10 to 50% reduction of the stiffness of element 4. This means that the truss changes its dynamical behaviour five times. Each value of the damage index, evaluated by introducing $r = 6$ into equation (3) and plotted in Figure 5, was obtained by simulating 10 measurements of the frequency response functions and by incorporating these data into matrix $\mathbf{M}$. Moreover, in order to

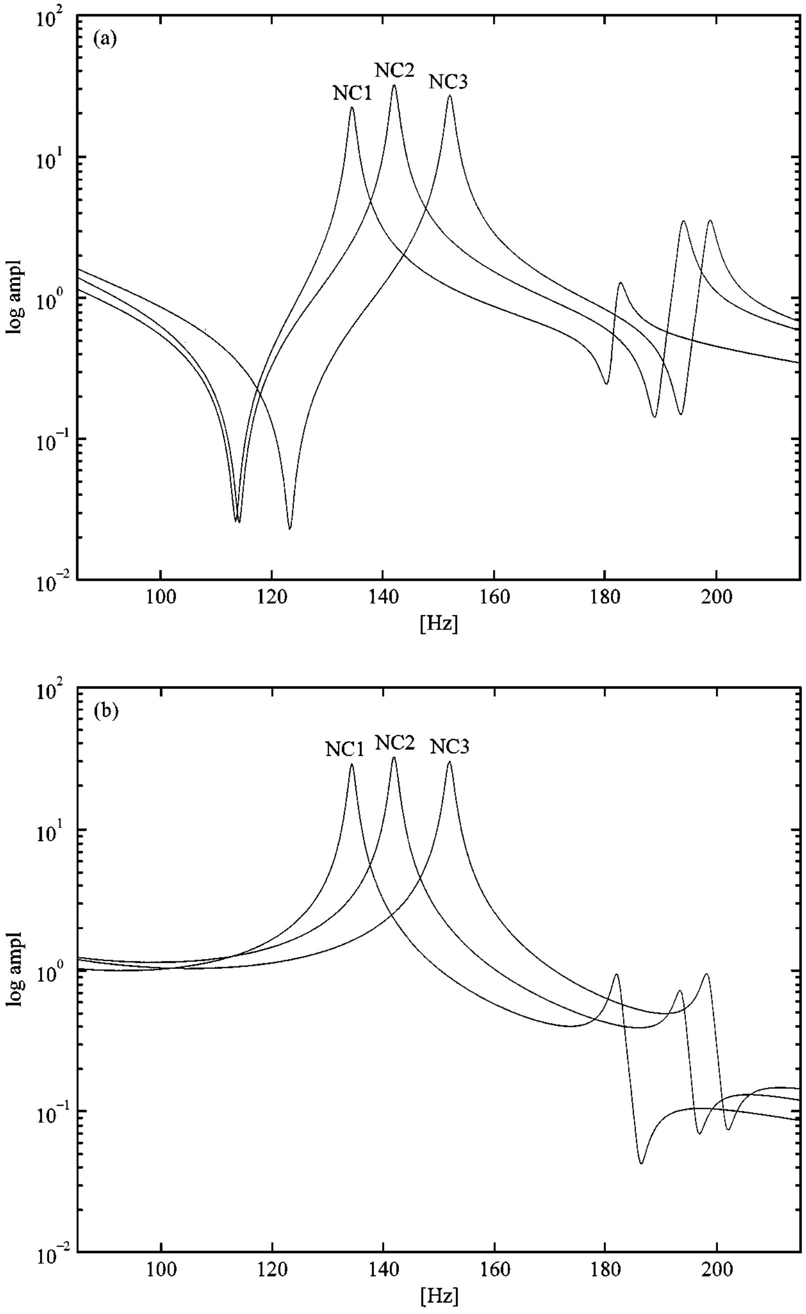

Figure 4(a). $H^{(1)}(\omega)$ related to the three undamaged operating conditions; (b) $H^{(2)}(\omega)$ related to the three undamaged operating conditions.

simulate the passing of time, the damage index was evaluated 10 times for each condition, i.e., a set of 10 measurements was analyzed.

Figure 5 highlights that the damage index is almost constant when the structure is undamaged, without any influence of the current condition. Furthermore, as

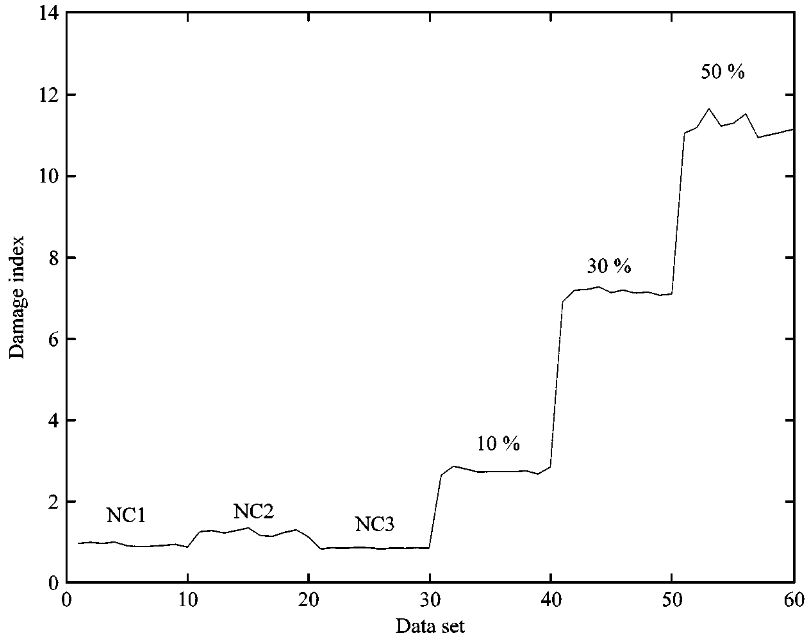

Figure 5. Damage index D evaluated for the three levels of damage related to condition 1.

soon as the structure undergoes a damage, which occurs for data set no. 31, the damage index increases enabling the fault to be detected.

3.2. ANALYSIS OF THE EXPERIMENTAL DATA

Experimental tests involving a cantilever steel beam of length 1 m and cross-section with width $b = 0{\cdot}02$ m and height $h = 0{\cdot}02$ m have been performed. In order to simulate different operating conditions during tests, two concentrated masses M_1 and M_2, with weights of 0·6 and 1·2 kg, respectively, have been connected to the beam in a position close to the free end, as shown in Figure 6. As a consequence, three conditions can be taken into account: NC1: concentrated masses not connected to the beam; NC2: the lighter mass connected close to the free end; NC3: the heavier mass connected close to the free end.

After producing a series of vibration tests related to the undamaged beam, a notch of width 1·5 mm was cut with an increasing depth from 1 to 4 mm with a step of 1 mm. Hence, it was possible to evaluate the damage index D for increasing notch depth.

Frequency response functions, related to the four sensors shown in Figure 6, were estimated 40 times for each undamaged condition, and 20 times for each damaged condition and each level of damage by exciting the structure at the free end with a random signal. FRFs related to the accelerometer located at the free end are shown in Figure 7 for all the three operating conditions and unnotched beam.

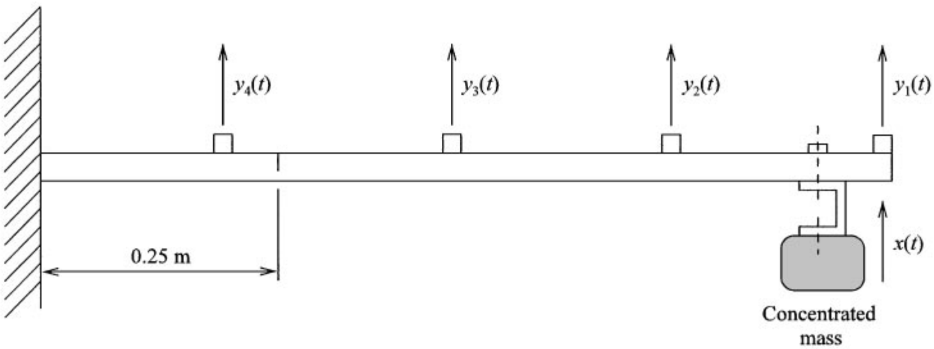

Figure 6. The beam under test.

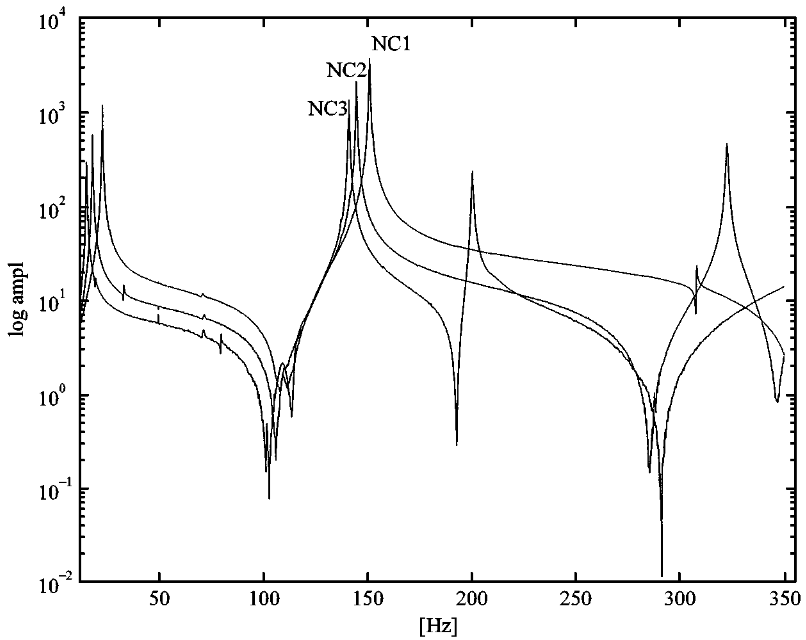

Figure 7. FRFs related to the three undamaged conditions and to the sensor at the free end.

As in the previous example, characteristic matrices were built by using the amplitude of FRFs from 12 to 175 Hz, which corresponds to 1114 samples in frequency, and related to the four sensors located on the beam. As a result, each $\mathbf{W}_i$ has four columns, of which just two are linearly independent such that $m = 2$ (Figure 7 shows that each operating condition has two modes in the analyzed frequency range). By collecting the characteristic matrices into matrix $\mathbf{M}$ for the three operating conditions, as in the previous example, it follows that the rank r of matrix $\mathbf{M}$ will be equal to 6 if the structure is undamaged. Then the damage index was evaluated 14 times for each condition both for damaged and undamaged states.

The trend of the index is shown in Figures 8(a)–(c) which corresponds to the damaged structure in the first, second and third operational conditions respectively. These figures also show that it is not possible to detect the notch with a height of 1 mm for conditions 2 and 3; this problem can be explained with reference to Figure 9

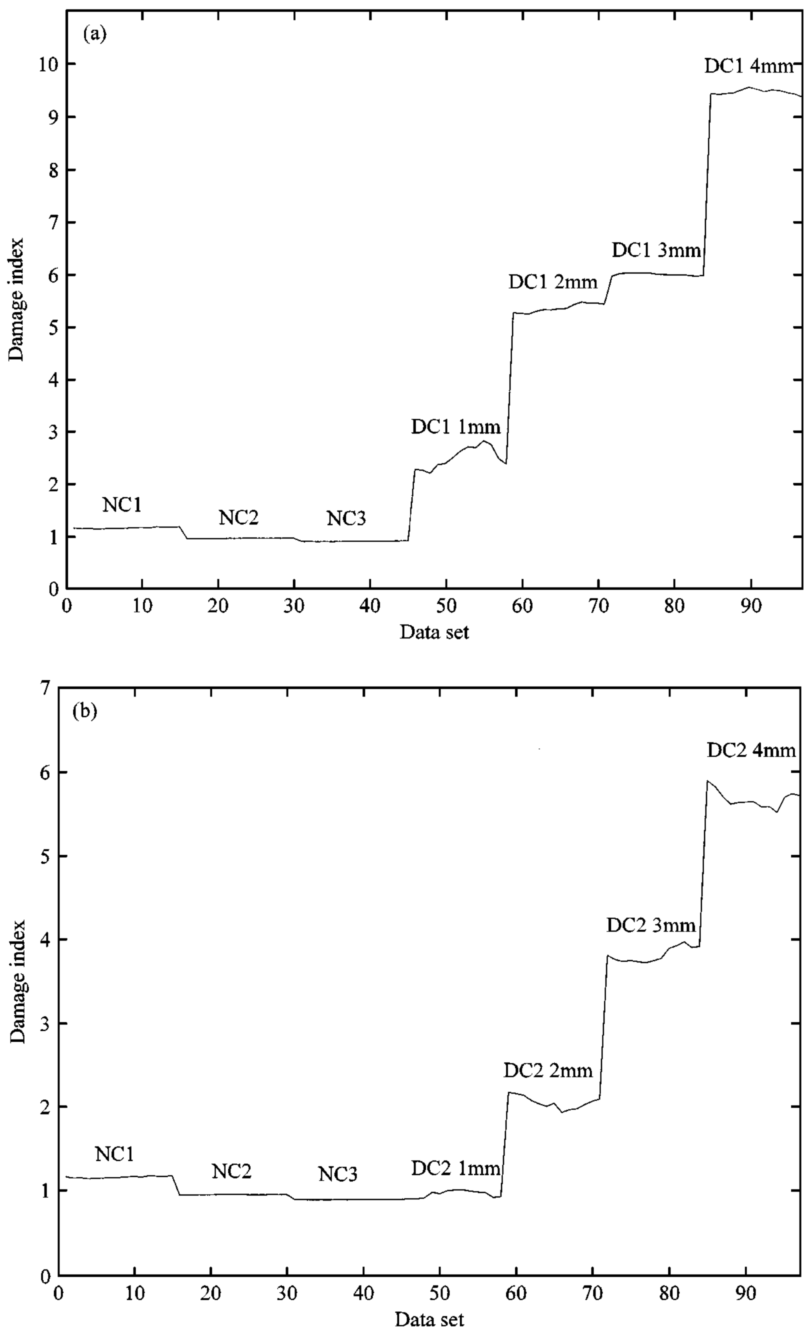

Figure 8. Index of damage for (a) condition no. 1; (b) condition no. 2; (c) condition no. 3.

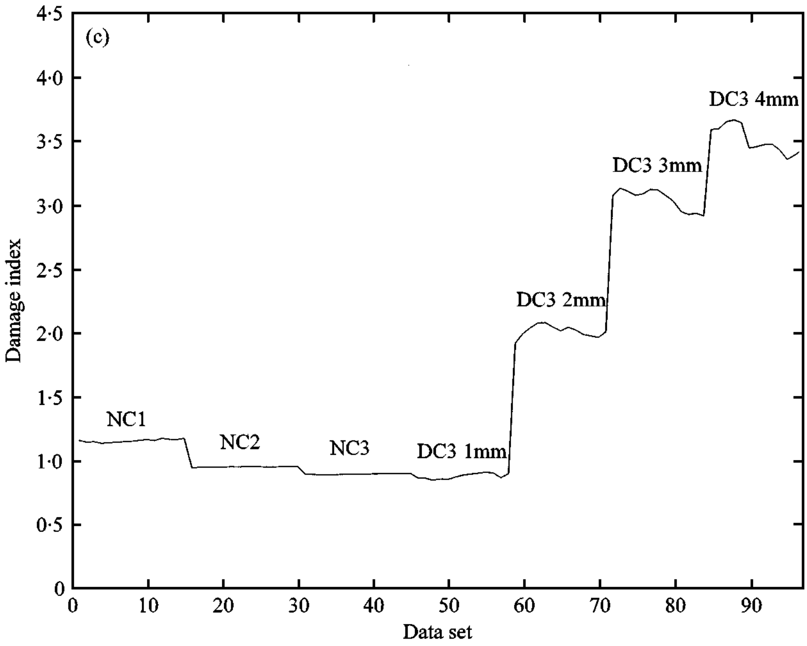

Figure 8. Continued.

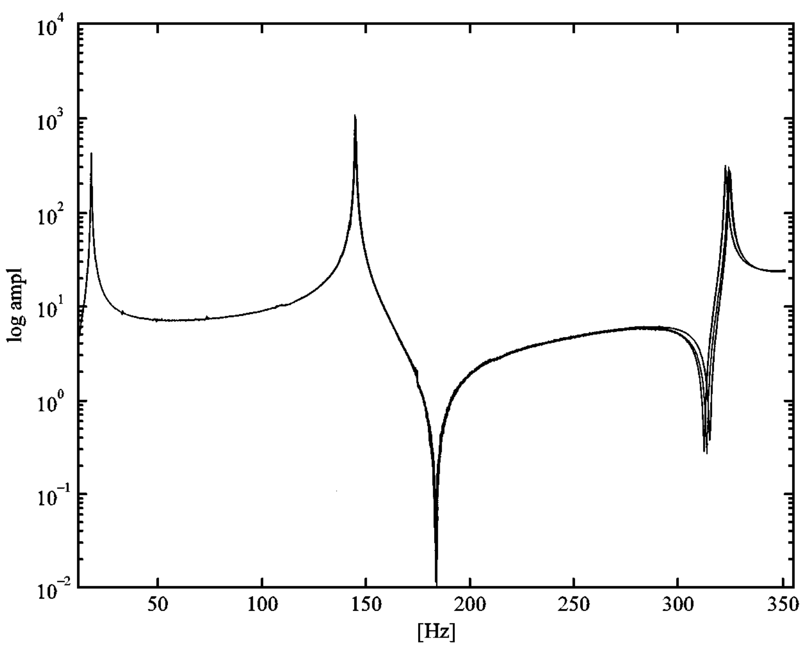

Figure 9. FRFs (sensor no. 2) for condition no. 2 both undamaged and damaged at the four levels.

in which FRFs for the second condition, in both undamaged and damaged states at various levels, are shown and from which it can be seen that it is difficult to distinguish any difference in the analyzed frequency range. Instead, if the frequency range from 120 to 350 Hz is used, damage can be detected, as shown in both Figures 10 and 11. The latter figure highlights the effect of the increase in singular values due to the presence of damage, as discussed in the previous sections.

4. CONCLUSIONS

The method presented in this paper addresses the task of damage detection even if the structure under test changes operational condition. The ability of this technique to detect damage has been demonstrated by analyzing numerical and experimental data and in both cases satisfactory results have been obtained.

The main advantage of this method over other techniques which permit the detection of damage in structures with different operational conditions lies in the straightforward way in which it can be applied. Indeed, it is sufficient to measure data which represent the response characteristics of the structure, even by using sensors of different types, and incorporate them in a matrix before evaluating the damage index proposed in section 2.3. Damage can be detected by following the trend of this index over time and identifying any sharp increase in the index which would correspond to the occurrence of damage.

Further research by the authors will be undertaken to enable for the method to deal with systems which alter their operational conditions continuously in time.

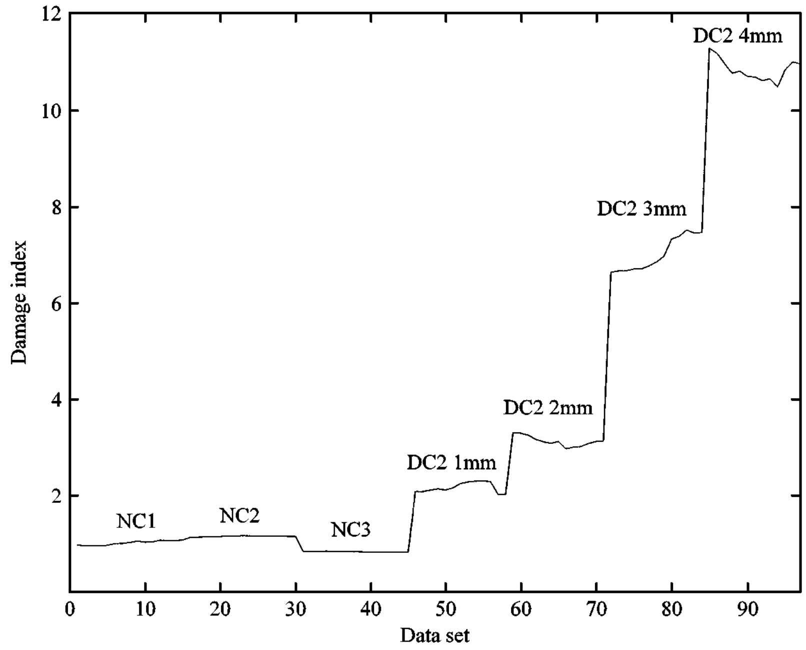

Figure 10. Index of damage for condition no. 2 (frequency range 120–350 Hz).

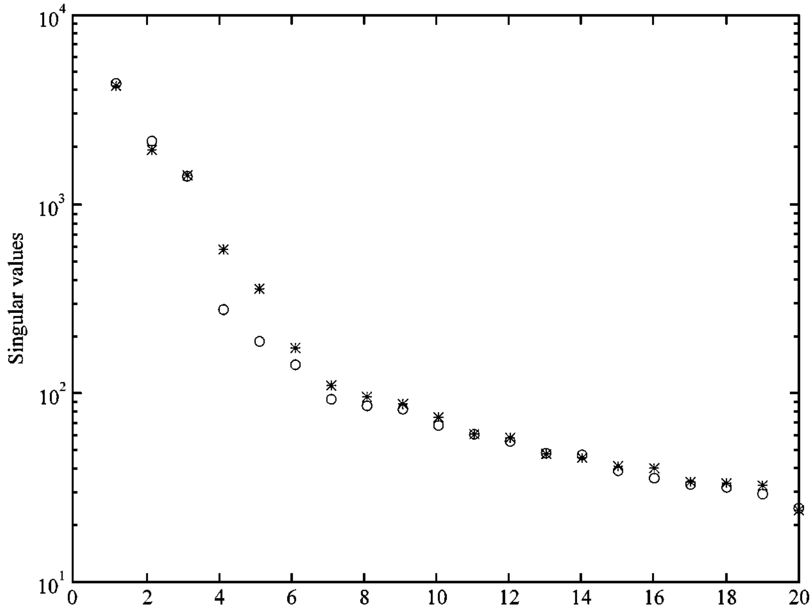

Figure 11. First 20 singular values in decreasing order related to undamaged (○) and damaged (at first level) (*) condition no. 2 (frequency range 120–350 Hz).

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